Final exam — Partial Differential Equations (WBMA008-05)

Friday 21 June 2024, 15.00h-17.00h

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a "cheat sheet" (one sheet A4, both sides, handwritten,"wet ink").
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 6 + 8 = 20 points)

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + e^{-x}\frac{\partial u}{\partial x} = 0, \quad u(0,x) = f(x).$$

(a) Compute all characteristic curves; express the answer as $x = x(t)$.

(b) Determine the region D of the (t,x)-plane in which the solution is determined by the initial condition.

(c) Compute the solution $u(t,x)$ for every $(t,x) \in D$.

Problem 2 (5 + 15 = 20 points)

Consider the following heat equation for $0 < x < 1$ and $t > 0$:

$$u_t = u_{xx}, \quad u_x(t,0) = u(t,0), \quad u_x(t,1) = -u(t,1).$$

(a) Use the anszatz $u(t,x) = e^{\lambda t}v(x)$ and derive a boundary value problem for v.

(b) Show that there exist infinitely many nontrivial solutions for $\lambda < 0$.

Problem 3 (10 points)

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that

$$u(-1,3) = 5 \quad \text{and} \quad u(x,y) = x \ \text{ when } \ (x+1)^2 + (y-3)^2 = 4.$$

Show that u is *not* harmonic.

Turn page for problems 4 and 5!

Problem 4 (12 + 8 = 20 points)

Recall the following function:

$$G_0(x,y;\xi,\eta) = -\frac{1}{2\pi}\log\|(x,y)-(\xi,\eta)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm.

(a) Use the method of images to construct Green's function for Poisson's equation on

$$\Omega = \{(x,y)\in\mathbb{R}^2 : y > 1\}.$$

(b) Compute the normal derivative along $\partial\Omega$ (with respect to the variables ξ and η) of Green's function constructed in part (a).

Problem 5 (20 points)

Use Fourier transforms to solve the following equation:

$$\frac{\partial^2 u}{\partial t\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = \frac{1}{x^2+9}.$$

Express the solution explicitly (i.e. without using integrals).

Please do not forget to complete the course evaluation!

End of test (90 points)

Solution of problem 1 (6 + 6 + 8 = 20 points)

(a) The characteristic curves are found by solving the equation $dx/dt = e^{-x}$.
(2 points)

By introducing the function

$$\beta(x) = \int e^x \, dx = e^x,$$

we can write the characteristic curves as

$$t \mapsto (t, \beta^{-1}(t+k)) = (t, \log(t+k)),$$

where $k \in \mathbb{R}$ is an arbitrary constant and $t > -k$.
(4 points)

(b) Along a characteristic curve the solution u is constant. To determine the value of this constant we need to use the initial condition and that is only possible when the characteristic curve intersects the x-axis.
(3 points)

Note that the characteristic curves intersect the x-axis if and only if $k > 0$. This means that the solution $u(t,x)$ is only determined by the initial condition in the region

$$D = \{(t,x) \in \mathbb{R}^2 \,:\, t \leq 0\} \cup \{(t,x) \in \mathbb{R}^2 \,:\, t > 0 \text{ and } x > \log(t)\}.$$

(3 points)

(c) *Method 1.* In the region D the solution is given by

$$u(t,x) = f(\beta^{-1}(\beta(x) - t)) = f(\log(e^x - t)).$$

(8 points)

Method 2. Assume that $(\bar{t}, \bar{x}) \in D$. Observe that this point lies on the characteristic curve with $k = e^{\bar{x}} - \bar{t}$. This curve intersects the x-axis in the point $(0, \log(e^{\bar{x}} - \bar{t}))$.
(4 points)

Since solutions are constant along the characteristic curve we have

$$u(\bar{t}, \bar{x}) = u(0, \log(e^{\bar{x}} - \bar{t})) = f(\log(e^{\bar{x}} - \bar{t})).$$

Dropping the bars from the notation gives the desired expression.
(4 points)

Solution of problem 2 (5 + 15 = 20 points)

(a) Substituting the ansatz $u(t,x) = e^{\lambda t} v(x)$ into the equation gives the following boundary value problem for the function v:

$$v''(x) - \lambda v(x) = 0, \quad v'(0) = v(0), \quad v'(1) = -v(1).$$

(5 points)

(b) For $\lambda = -\omega^2$ with $\omega > 0$ we have $v(x) = a\cos(\omega x) + b\sin(\omega x)$.
(4 points)

The boundary conditions imply that

$$\begin{aligned} b\omega &= a, \\ -a\omega\sin(\omega) + b\omega\cos(\omega) &= -a\cos(\omega) - b\sin(\omega). \end{aligned}$$

(4 points)

Substituting the first equation into the second gives

$$b\big[(1-\omega^2)\sin(\omega) + 2\omega\cos(\omega)\big] = 0.$$

(Alternatively, we can find the expression in square brackets by computing the determinant of the coefficient matrix.)
(4 points)

For a nontrivial solution we need $b \neq 0$ and thus

$$\tan(\omega) = \frac{2\omega}{\omega^2 - 1}.$$

(4 points)

Note that the right hand side tends to zero as $\omega \to \infty$. Since the tangent is π-periodic, the above equation has countably many solutions.
(3 points)

Solution of problem 3 (10 points)

Method 1: using the maximum principle. Let D the be disc with center $(-1,3)$ and radius 2. If u is harmonic on this disc, then it follows that the maximum and minimum values of u can only be attained on the boundary of D.
(4 points)

It is clear that the maximum value of u on ∂D is given by the maximum x-coordinate of points along ∂D. The maximum value is given by $-1+2=1$.
(2 points)

However, the value of u at the center of D is larger. Indeed, $u(-1,3)=5>1$. Therefore, the maximum value of u is attained in the interior of D which contradicts the maximum principle. We conclude that u cannot be harmonic.
(4 points)

Method 2: using the mean value property. Let D the be disc with center $(-1,3)$ and radius 2. If u is harmonic, then the mean value property gives

$$u(-1,3)=\frac{1}{4\pi}\oint_{\partial D} u\,ds.$$

(3 points)

Computing the line integral on the right hand side gives

$$\begin{aligned}\frac{1}{4\pi}\oint_{\partial D} u\,ds &= \frac{1}{4\pi}\int_0^{2\pi} u(-1+2\cos t, 3+2\sin t)2\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} -1+2\cos t\,dt\\ &= \frac{1}{2\pi}\left[-t+2\sin t\right]_0^{2\pi}\\ &= -1.\end{aligned}$$

(5 points)

So the mean value property gives the contradiction $5=-1$. From this we conclude that u cannot be harmonic.
(2 points)

Solution of problem 4 (12 + 8 = 20 points)

(a) We construct the Green's function by setting $G = G_0 + z$, where the function z satisfies $\Delta z = 0$ on Ω and $z = -G_0$ on $\partial\Omega$. To a point $(\xi,\eta) \in \Omega$ we associate an image point $(\xi',\eta') \in \mathbb{R}^2 \setminus \overline{\Omega}$. The ansatz

$$z(x,y;\xi,\eta) = \frac{a}{2\pi}\log\|(x,y)-(\xi',\eta')\| + \frac{b}{2\pi}.$$

guarantees that z is harmonic on Ω.
(3 points)

Take (ξ',η') to be the reflection of (ξ,η) through the line $y = 1$:

$$(\xi',\eta') = (\xi, 2-\eta).$$

Then for all points $(x,1) \in \partial\Omega$ we have

$$\|(x,1)-(\xi,\eta)\| = \|(x,1)-(\xi,2-\eta)\|.$$

(6 points)

Setting $a = 1$ and $b = 0$ gives Green's function:

$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\log\|(x,y)-(\xi,\eta)\| + \frac{1}{2\pi}\log\|(x,y)-(\xi,2-\eta)\|.$$

(3 points)

Equivalently, we can write

$$G(x,y;\xi,\eta) = -\frac{1}{4\pi}\log\left[(x-\xi)^2+(y-\eta)^2\right] + \frac{1}{4\pi}\log\left[(x-\xi)^2+(y-2+\eta)^2\right].$$

(b) At any point $(\xi,1) \in \partial\Omega$ the outward normal unit vector is given by $\mathbf{n} = (0,-1)$, and thus

$$\frac{\partial G}{\partial \mathbf{n}}(x,y;\xi,1) = (\nabla G \bullet \mathbf{n})(x,y;\xi,1) = -\frac{\partial G}{\partial \eta}(x,y;\xi,1)$$

(2 points)

We have that

$$-\frac{\partial G}{\partial \eta}(x,y;\xi,\eta) = -\frac{1}{2\pi}\cdot\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2} - \frac{1}{2\pi}\cdot\frac{y-2+\eta}{(x-\xi)^2+(y-2+\eta)^2}$$

(4 points)

Finally, evaluating for $\eta = 1$ gives

$$\begin{aligned}-\frac{\partial G}{\partial \eta}(x,y;\xi,1) &= -\frac{1}{2\pi}\cdot\frac{y-1}{(x-\xi)^2+(y-1)^2} - \frac{1}{2\pi}\cdot\frac{y-1}{(x-\xi)^2+(y-1)^2}\\ &= -\frac{1}{\pi}\cdot\frac{y-1}{(x-\xi)^2+(y-1)^2}.\end{aligned}$$

(2 points)

Solution of problem 5 (20 points)

Taking the Fourier transform of the equation gives

$$ik\frac{d\widehat{u}}{dt} = (ik)^2\widehat{u} \quad \text{and thus} \quad \frac{d\widehat{u}}{dt} = ik\widehat{u}.$$

(3 points)

The solution of this equation is given by

$$\widehat{u}(t,k) = \widehat{u}(0,k)e^{ikt}.$$

(2 points)

From the list of Fourier transforms we obtain:

$$\mathscr{F}\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2} \quad \text{where } a > 0.$$

(5 points)

Setting $a = 3$ and using the symmetry principle gives

$$\widehat{u}(0,k) = \mathscr{F}\left[\frac{1}{x^2+9}\right] = \frac{1}{3}\sqrt{\frac{\pi}{2}}e^{-3|-k|} = \frac{1}{3}\sqrt{\frac{\pi}{2}}e^{-3|k|}.$$

(5 points)

In conclusion, we have

$$\widehat{u}(t,k) = \frac{1}{3}\sqrt{\frac{\pi}{2}}e^{-3|k|}e^{ikt}.$$

Note that by the shift theorem the factor e^{ikt} results in replacing x by $x+t$ after taking the inverse Fourier transform. This gives:

$$u(t,x) = \frac{1}{(x+t)^2+9}.$$

(5 points)